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Minimum Time Pulse Response Based Control of Flexible Structures

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This paper presents the pulse response based control method for minimum time control of structures. An explicit model of a structure is not needed for this method, because the structure is represented in terms of its measured response to pulses in control inputs. Minimum time control problems are solved by finding the minimal number of time steps for which a control history exists that consists of a train of pulses, satisfies input bounds, and results in desired outputs at the end of the control task. There is no modal truncation in the pulse response representation of the response, because all modes contribute to the pulse response data. The precision with which the final state of the system can be specified using pulse response based control is limited only by the observability of the system with the given set of outputs. A special algorithm for solving the numerical optimization problem arising in pulse response based control is presented, and the effect of measurement noise on the accuracy of the final predicted outputs is investigated. A numerical example demonstrates the effectiveness of pulse response based control and the algorithm used to implement it. The pulse response based control method is applied to linear problems in this paper.

Introduction

IN minimum time maneuver problems for flexible spacecraft, bounded control inputs are used to carry out a prescribed change in rigid body modal states as quickly as possible, with minimal residual vibration at the end of the maneuver. Because flexible spacecraft are distributed parameter systems, discrete models must be of a relatively high order to represent their behavior accurately. For practical problems, this leads to formidable difficulties associated with both system identification and numerical solution of optimal control problems.

Over the past decade or so, a number of different approaches have been taken to address these problems, and some of these are briefly reviewed here. Turner and Junkins[1] represent flexible motion in terms of several assumed modes, and minimize a quadratic performance index with a single-actuator control, with specified final states and final time. A continuation method is used to obtain a solution of the nonlinear two-point boundary value problem that results when kinematic nonlinearity is present in the formulation. Breakwell[2] uses a standard linear-quadratic regulator approach to achieve prescribed final states in a finite set of modes of vibration at a prescribed final time. Turner and Chun[3] extend the approach of Turner and Junkins for the case in which a number of actuators are distributed throughout the structure. Chun et al.[4] obtain a frequency-shaped open-loop control for the rigid body modes, using a continuation method to handle nonlinearity, and then design a feedback control for the flexible motion by linearizing the flexible response about several points in the open-loop rigid body trajectory. In a later paper, they replace the solution of the open-loop problem for the rigid body modes with a programmed-motion/inverse dynamics approach, where the trajectories of the rigid body modes are simply specified as smooth functions and the required control torques are obtained from the nonlinear rigid body equations of motion.[5] Baruh and Silverberg[6] present a method for addressing maneuver and vibration suppression independently in a linearized flexible maneuver problem. Meirovitch and Quinn[7] develop an approach in which the rigid body maneuver is taken as a zero order problem and the flexible motion is treated as a first order perturbation for nonlinear flexible maneuver problems. Meirovitch and Sharony[8] then use this perturbation approach and obtain a minimum time (bang-bang) solution of the rigid body problem, and a linear quadratic regulator with integral feedback and prescribed convergence rate to drive the flexible motion to zero at the end of the maneuver. Bounds on control inputs are only considered in the design of the rigid body control. True minimum time control is investigated by Ben-Asher et al.,[9] in which the system is modeled by the assumed modes method and switching times for the bang-bang control are found for linear and nonlinear problems using parameter optimization, and by Singh et al.,[10] in which assumed modes are used, and a set of nonlinear algebraic equations are solved for switching times using a homotopy method.

All of these methods rely on an explicit model of the spacecraft, which makes accurate system identification a necessity. This requirement presents difficulties for spacecraft with high modal density. The model will have to be truncated to satisfy computational constraints, while ensuring that unmodeled dynamics will not be significant in the response of the spacecraft. Verification of the model is complicated by the need to compensate for gravitational and atmospheric effects if testing is done on the Earth's surface, or by limitations on test hardware available in orbit. Time-varying systems present the additional complication that their models must be updated and verified after any significant change in properties.

Another issue associated with finite-order modeling of flexible spacecraft is that, although a controllable finite-order model of a distributed parameter system is guaranteed to have a bang-bang minimum time control,[11] the minimum time control of the class of distributed systems that includes flexible spacecraft is not necessarily bang-bang, as proved by Balakrishnan in 1965.[12] Indeed, the only known exact solution for minimum time control of a flexible structure by means of bounded inputs is not bang-bang.[13] For this reason, if minimum time control of a distributed system, rather than of its finite-order model, is of interest, it is not appropriate to require control profiles to be bang-bang.

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This paper presents a new approach to minimum time control of flexible spacecraft that is not subject to the difficulties mentioned previously. The flexible behavior of the spacecraft is represented in terms of measured response to pulses in control inputs, rather than in terms of an explicit model. A minimum time control profile is obtained that results in prescribed outputs at the end of the control task, according to convolution of the pulse response data. The main advantage of this approach is that measurements of pulse response can be obtained much more easily and directly than the explicit model data required for conventional approaches. There is no modal truncation because all modes excited by the control inputs participate in the pulse response data, assuming that the system is observable. The present paper addresses application of Pulse Response Based Control (PRBC) to linear problems.

In the second section of this paper, the PRBC method is presented. The third section presents an efficient numerical algorithm for implementing PRBC. The effect of measurement noise on the accuracy of the final state of the system is investigated in the fourth section. In the fifth section, a numerical example is presented. The final section contains conclusions.

Pulse Response Based Control

Suppose a linear dynamic system is controlled by a single input $u(t)$, with no other excitations, and its response is measured in terms of m outputs that constitute a vector $y(t)$. If the system is initially at rest and a square pulse in $u(t)$, of unit amplitude and duration Δt, is applied, and the outputs $y(t)$ are sampled every Δt for a total time $n\Delta t$, an $m \times n$ pulse response matrix H is obtained, of the form

$$\begin{aligned} H &\equiv \{y(n\Delta t) \quad y[(n-1)\Delta t] \quad \cdots \quad y(\Delta \tau)\} \\ &= [h_1 \quad h_2 \quad \cdots \quad h_n] \end{aligned} \tag{1}$$

where the order of the output vectors is reversed for use in a convolution sum. Suppose the system is initially at rest and a control profile $u(t)$ that is constant over each time step Δt between $t = 0$ and $t = n\Delta t$ is applied to the system, so that the control input is equal to u_i over the ith time step. Then the output vector at $n\Delta t$ can be found by convolution because of the linearity of the system. Each value u_i is multiplied by a vector in the pulse response matrix H, so that the output vector is given by

$$y(n\Delta t) = \sum_{i=1}^{n} h_i u_i = Hu \tag{2}$$

where $u = [u_1 \quad u_2 \quad \cdots \quad u_n]^T$.

If there are p inputs, the scalar entries u_i in the vector u are replaced by the subvectors $u_i = [u_{1,i} \quad \cdots \quad u_{p,i}]^T$, and the vectors h_i must be replaced by $m \times p$ submatrices H_i. The pulse response matrix H is then filled by applying pulses in one control input at a time, and sampling the outputs resulting from each pulse every Δt over an interval of length $n\Delta t$. The outputs resulting from a pulse in the first control input will fill the first vector of each submatrix H_i in H, and each of the other $p-1$ columns in a submatrix H_i will be obtained from response to a pulse in another control input.

If a control task that consists of driving the system from rest at $t = 0$ to a desired state at a time $t_f = n\Delta t$ is successfully accomplished by a piecewise constant control profile of the type described earlier, the set of linear equations

$$y(t_f) = Hu \tag{3}$$

must be satisfied by the control profile u, where the vector $y(t_f)$ contains outputs consistent with the desired final state of the system. Also, if there is no excitation of the system after t_f, the equations

$$y(t_f + j\Delta t) = [H_{1-j} \quad H_{2-j} \quad \cdots \quad H_{n-j}]u \tag{4}$$

must be satisfied, where the time-shifted submatrix H_{k-j} contains outputs measured $(n - k + 1) + j$ time steps after applying test pulses in control inputs, and where the outputs $y(t_f + j\Delta t)$ must be consistent with the desired state of the system at t_f, with free response after t_f.

These observations suggest the following approach for minimum time control problems: Find the smallest integer number of time steps n such that there exists a control history vector $u \equiv [u_1^T \cdots u_n^T]^T$ satisfying the equations

$$\bar{H}u = \bar{y}_f \tag{5}$$

and the input bounds, typically of the form

$$B_j^l \le u_{j,i} \le B_j^u, \qquad j = 1, \ldots, p; \qquad i = 1, \ldots, n \tag{6}$$

in which B_j^l and B_j^u are lower and upper bounds for the jth control input. The vector $\bar{y}_f$ contains outputs at t_f and at l time steps afterward:

$$\bar{y}_f \equiv \begin{Bmatrix} y(t_f) \\ y(t_f + \Delta t) \\ \vdots \\ y(t_f + l\Delta t) \end{Bmatrix} \tag{7}$$

and the matrix $\bar{H}$ is given by

$$\bar{H} \equiv \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_0 & H_1 & \cdots & H_{n-1} \\ \vdots & \vdots & & \vdots \\ H_{1-l} & H_{2-l} & \cdots & H_{n-l} \end{bmatrix} \tag{8}$$

Satisfaction of Eq. (5) is a necessary but not sufficient condition for accomplishing the control task exactly. An important question is how precisely the final state of the system is specified by these equations. If the system is of finite order with state equations in the standard form

$$\dot{x} = Ax + Bu \tag{9}$$

where x is the state vector and A and B are constant, and the outputs are related to the states by the equation $y = Cx$, then the vector $\bar{y}_f$ is given by

$$\bar{y}_f = \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^l \end{bmatrix} x(t_f) \tag{10}$$

where $\Phi = e^{A\Delta t}$ is the state transition matrix for the time step Δt. It is apparent from this that if a finite-order system is observable by means of the outputs in y, then the final state can be specified exactly by this approach. In general, if there are s states and the rank of $\bar{H}$ is r, then the final state $x(t_f)$ is confined to a subspace of order $s - r$.

Flexible spacecraft are distributed parameter systems of infinite order. However, practical considerations ordinarily require finite-order representation for control, typically using a truncated modal model. The PRBC approach outlined earlier, on the other hand, can be used on linear distributed parameter systems without any explicit modeling. In addition, because all modes can participate in the pulse response measurements, there is no modal truncation in the system representation used by PRBC. By specifying the outputs at the final time t_f and at l time steps afterward, PRBC places a finite set of constraints on the final state, as modal methods do when the final states of a finite number of modes are specified. An important difference between the two is that there is no assumption of finite-order dynamics for the system in PRBC.

If actuators do not respond instantaneously to control commands, but instead exhibit linear response behavior, the actuator dynamics can be considered part of the system dynamics. Square pulses in control commands, rather than in the actual forces or moments applied to the system, are used to obtain pulse response data, and the vector $\boldsymbol{u}$ represents command profiles rather than force or moment profiles.

The outputs in y will typically include measurements of response quantities that are most critical to the performance of the spacecraft. Therefore, specifying these outputs will have the natural effect of specifying the final states of modes that are most important to performance, without explicit consideration of which modes are most important. In comparison with the conventional modal approach, which requires explicit identification of modes from response measurements and explicit modeling of response in these modes, PRBC is a more direct approach.

Ordinarily the inertial properties of spacecraft are known with a high level of accuracy, especially in comparison with modal properties. If this is the case, it is straightforward to determine the response of a structure in its rigid body modes to pulses in the control inputs, so that the desired rigid body mode states at the final time t_f can be included in the vector of specified outputs $\bar{y}_f$. Of course, if pulse responses are known accurately for any flexible modes, their final modal states can also be included in $\bar{y}_f$.

The matrix $\bar{H}$ is obtained by simply collecting measurements of pulse response, and it will not necessarily be of full rank. This can lead to numerical difficulties in the solution of the optimization problem, because the equality constraints will not be linearly independent if $\bar{H}$ is rank deficient. This problem can be easily handled by finding the singular value decomposition of $\bar{H}$, so that

$$\bar{H} = U\Sigma V^T \tag{11}$$

where U and V are square orthogonal matrices containing the left and right singular vectors of $\bar{H}$, and Σ is a rectangular matrix consisting of a diagonal square submatrix containing singular values of $\bar{H}$, with null columns appended on the right to make the matrices conformal. If a tolerance ϵ is chosen so that singular values less than ϵ are taken to indicate rank-deficiency, then the matrices $\tilde{U}$, $\tilde{\Sigma}$, and $\tilde{V}$ can be obtained from U, Σ, and V by retaining only the columns associated with singular values greater than ϵ. A maximal set of linearly independent equality constraints that agree with the original, possibly dependent, equality constraints can be written as

$$\tilde{\Sigma}\tilde{V}^T\boldsymbol{u} = \tilde{U}^T\bar{y}_f \tag{12}$$

Even if the original equality constraints are linearly independent, this step can improve numerical conditioning because of the orthogonality of the columns of $\tilde{V}$.

It is worth noting that since the matrix $\bar{H}$ is a generalized Hankel matrix, the maximum rank of $\bar{H}$ that can be obtained by adding rows corresponding to outputs at additional time steps is equal to the maximum order of a system model that can be realized from the outputs in y.[14]

Algorithm for the Optimization Problem

The PRBC method addresses the minimum time control problem by finding the minimum number of time steps for which a control profile exists satisfying the equality constraints of Eq. (5) and the input bounds. The resulting numerical optimization problem is therefore an integer programming problem. In this section, an algorithm is presented for solving this integer programming problem using standard linear programming methods.

Before the problem is solved for a given control task and time step size Δt, the number of time steps for which outputs must be specified in $\bar{y}_f$, to adequately specify the desired final state of the system, is not known. If this and the minimum time required for control t_f were known in advance, a piecewise constant approximation of the minimum time control profile could be obtained by simply finding a vector $\boldsymbol{u}$ satisfying the equality constraints and the input bounds, with a number of time steps n in the control profile such that $n\Delta t \geq t_f$. To verify that n is a minimal number of time steps, it is only necessary to show that no feasible solution exists with $n - 1$ time steps.

However, both t_f and the number of time steps for which outputs must be specified are not known in advance. Perhaps the most straightforward approach to the problem is to begin by including only the final rigid body mode states and the outputs at the final time in $\bar{y}_f$, and incrementing the number of time steps in the control history until a feasible solution $\boldsymbol{u}$ can be found. Then the desired outputs for $t_f + \Delta t$ can be added to $\bar{y}_f$, and the number of time steps n can be increased again until a feasible solution is found. When including outputs for more time steps in $\bar{y}_f$ no longer causes the number of time steps in the control history or the control profile $\boldsymbol{u}$ to change, it can be concluded that a control profile that achieves the desired final state in a minimal number of time steps has been found. The disadvantage of this approach is its computational expense. If the control profile is to be accurate, the time step Δt must be small. This condition implies that there will be a large number of unknowns in the vector $\boldsymbol{u}$, so that investigating existence of feasible solutions will be costly, especially if it must be done for many values of n.

The number of values of n for which existence of feasible solutions is investigated can be reduced by getting an inexpensive estimate of t_f with a larger time step size and therefore fewer unknowns in the vector $\boldsymbol{u}$. Once pulse response data for pulses of width Δt has been obtained in a matrix $H^{\Delta t}$, it is straightforward to obtain a pulse response matrix for pulses of width $q\Delta t$, where q is an integer:

$$H^{q\Delta t} \equiv [H_1^{q\Delta t} \; H_2^{q\Delta t} \; \cdots] = \left[\sum_{i=1}^{q} H_i^{\Delta t} \; \sum_{i=q+1}^{2q} H_i^{\Delta t} \; \cdots\right] \tag{13}$$

Then each entry in a corresponding $\boldsymbol{u}$ vector gives the value of a control input over a time interval of length $q\Delta t$. The approach taken in the algorithm presented here is to recursively generate pulse response matrices $H^{2\Delta t}$, $H^{4\Delta t}$, $H^{8\Delta t}$, etc., by adding consecutive pairs of submatrices H_i for one step size to obtain submatrices for a step size twice as large. Then the minimum time problem is solved first with the largest step size, and this solution gives a starting point for solving the problem with a step size half as large. The problem is solved with successively smaller step sizes until the original time step size Δt is reached.

There must be a criterion for determining when the control problem has been solved accurately enough to change to a smaller step size or ultimately to consider the problem solved. As an indicator of how accurately a given control profile causes the system to reach the desired final state, using only information available from pulse response data, the outputs for several time steps after the end of the control history, for which outputs have not been specified, are obtained by convolution. This is done for the first time, for reference, with the first minimum time control profile found, for which only the final rigid body mode states are specified and the largest time step size ΔT_0 is used. A vector $(\bar{e}_{\text{unsp}})_0$ is defined as

$$(\bar{e}_{\text{unsp}})_0 \equiv \begin{Bmatrix} e(t_f) \\ e(t_f + \Delta T_0) \\ \vdots \\ e(t_f + k\Delta T_0) \end{Bmatrix} \tag{14}$$

where each vector e is the error in the output vector y, k is an integer, and t_f is the final time obtained for this profile. For each new control profile obtained in the algorithm, a vector

$$\bar{e}_{\text{unsp}} \equiv \begin{Bmatrix} e(t_{\text{unsp}}) \\ e(t_{\text{unsp}} + \Delta T_0) \\ \vdots \\ e(t_{\text{unsp}} + k\Delta T_0) \end{Bmatrix} \tag{15}$$

is obtained, where $t_{\text{unsp}} = t_{\text{last}} + \Delta T_0$, where t_{last} is the last time step for which outputs have been specified for the current profile. The ratio of norms of these two vectors

$$R \equiv \frac{\|\bar{e}_{\text{unsp}}\|_2}{\|(\bar{e}_{\text{unsp}})_0\|_2} \tag{16}$$

is used as an indicator of how accurately the desired final state has been achieved. In principle, if the control task is accomplished exactly, there will be no error in outputs at any time after the end of the control history. In implementation of the algorithm, allowable values for R must be specified for each change in time step size and for final termination of the computation.

The following is an outline of the algorithm for solving the numerical optimization problem encountered in PRBC:

1) Obtain pulse response matrix H for time step Δt, from measurements of the system response. Also, given the system's inertial properties, calculate the matrix of pulse response in rigid body mode states H_R, for which the number of rows will be equal to the number of specified rigid body mode states.

2) From these matrices $H^{\Delta t}$ and $H_R^{\Delta t}$, obtain $H^{2\Delta t}$, $H^{4\Delta t}$, etc., and $H_R^{2\Delta t}$, $H_R^{4\Delta t}$, etc., as described earlier. Set ΔT equal to the largest time step size for which these matrices are generated.

3) First, solve the problem for the time step size ΔT considering only rigid body mode states.

a) Choose a number of time steps $n \geq (t_f)_{\text{RIGID}}/\Delta T$, where $(t_f)_{\text{RIGID}}$ is the minimum time interval in which the desired change in rigid body mode states can be accomplished with a set of p ideal control inputs having the given input bounds. If actuator dynamics are significant, n should be increased accordingly, but it is assumed for simplicity that n is chosen so that it is not possible to achieve the final rigid body mode states with fewer than n time steps.

b) Choose an initial guess u_0 appropriate for the control task at hand. For example, a bang-bang profile is likely to be a suitable choice for a rest-to-rest maneuver. With the desired final rigid body mode states in the vector y_R, obtain the error resulting from the initial guess as $e = y_R - H_R u_0$. Solve an auxiliary problem to determine whether a feasible solution exists with n time steps, by minimizing $J = w$, subject to the equality constraints

$$[H_R^{\Delta T} \quad e] \begin{Bmatrix} u \\ w \end{Bmatrix} = y_R \tag{17}$$

the input bounds, and the inequality $w \geq 0$, where w is an artificial variable. The upper-bounded simplex method[15] is appropriate since the unknowns are subject to both upper and lower bounds. If J cannot be driven to zero, the number of time steps must be increased. If $J = 0$, a control profile achieving the desired final rigid body mode states has been found. This profile will be modified over the course of the algorithm until the final solution is obtained. The norm of the error in unspecified outputs at the final time $n\Delta T$ and at several later time steps is calculated for comparison later in the algorithm.

4) Solve minimum time problems with outputs specified for an increasing number of time steps at the end of the control profile.

a) So that outputs can be specified at the final time, set

$$\bar{H} = \begin{bmatrix} H_R^{\Delta T} \\ H^{\Delta T} \end{bmatrix}$$

Find a control profile resulting in the desired rigid body mode states and outputs at the final time, if one exists, by computing $e = \bar{y}_f - \bar{H}u$, where $\bar{y}_f$ now contains both the final rigid body mode states and the outputs at the final time, and solving an auxiliary problem as in step 3b. Note that e is nonzero only in entries corresponding to newly applied constraints. Also, in the implementation of the simplex method, after each iteration it is only possible for these new constraints to be violated, since the equation analogous to Eq. (17) will always be satisfied, with monotonically decreasing values of w.

In general, outputs will be specified at the first time step of length ΔT at the end of the control history for which outputs have not yet been specified. Then the matrix $\bar{H}$ will take the form

$$\bar{H} = \begin{bmatrix} H_{R1}^{\Delta T} & \cdots & H_{Rn}^{\Delta T} \\ H_1^{\Delta T} & \cdots & H_n^{\Delta T} \\ H_0^{\Delta T} & \cdots & H_{n-1}^{\Delta T} \\ \vdots & \cdots & \vdots \end{bmatrix}$$

with a new row added each time outputs are specified for an additional time step. Similarly, a new subvector of outputs will be appended to $\bar{y}_f$ each time outputs are specified for another time step.

b) If $J = w$ for the auxiliary problem cannot be driven to zero, the number of time steps n must be increased by one. If $J = 0$, compute unspecified outputs at time steps at the end of the control profile, and check whether the ratio R is less than a value specified by the user for the current step size. If not, specify outputs for another time step, find a feasible control profile, and check again. If so, decrease the time step size by half.

5) To continue with a time step of half the size, set $\Delta T = \Delta T/2$ and $n = 2n$. The current control profile u is converted for the smaller step size by inserting, after the subvector for each time step, a duplicate copy of that subvector. Converting u for the smaller step size doubles the number of entries in u that are not equal to one of the bounds, so half of these that are closest to bounds are made equal to bounds, and the rest are retained as basic variables. The equation $\bar{H}u = \bar{y}_f$ is then solved for the basic variables. If any of these are found to violate a bound, a small auxiliary problem, involving only those variables that were not equal to bounds upon changing to a smaller step size, is solved to obtain a basic feasible solution.

When the time step size has been halved, the existence of a control profile satisfying the same set of equality constraints and the input bounds, and having fewer time steps than the new value of n, must be investigated. Frequently the final time can be reduced due to the ability to resolve the control profile more accurately with a smaller time step. The existence of a feasible control profile having $n - 1$ time steps is investigated by determining whether the control inputs for the first time step can be driven to zero in a feasible solution. For each positive (negative) entry $u_{j,1}$ in u_1, the lower (upper) bound is reset to zero, and a coefficient c_j is set to (minus) one. Then the objective function

$$J = \sum_{j=1}^{p} c_j u_{j,1}$$

is minimized subject to the equality constraints and the input bounds. If $J = 0$, a control profile with $n - 1$ steps has been found, and the existence of a control profile with one fewer time steps is investigated. When $J_{\min} \neq 0$, a minimal number of time steps has been found for which a feasible control profile with the time step size ΔT exists. At this point, the number of time steps for which outputs are specified is again increased, starting with the time $t_f + \Delta T$, since outputs have already been specified for t_f, $t_f + 2\Delta T$, $t_f + 4\Delta T$, etc. This continues until the ratio R decreases below the value specified for the current step size. Then the time step size is halved again.

Sampling for the ratio of output error norms R should be done each time with the largest step size ΔT_0 that is used at first, so that even though the step size is halved repeatedly, the length of time over which the system response is monitored in R remains consistent.

6) When the time step size Δt is reached, the minimum time control profile can be approximated with the maximum accuracy permitted by the pulse response data. The number of time steps for which outputs are specified is increased until the ratio R decreases below the value specified for the final result. At this point, the final time should have stopped increasing with the specification of additional outputs, since the desired final state of the system should be obtained very accurately. Since the control inputs are required by the simplex method to be equal to bounds for as many entries in $\boldsymbol{u}$ as possible, the control profile can usually be improved, in terms of the control effort expended and the smoothness of the profile, by minimizing the quadratic objective function $J = \boldsymbol{u}^T\boldsymbol{u}$ subject to the equality constraints $\bar{H}\boldsymbol{u} = \bar{y}_f$ and the input bounds, with the same number of time steps.

The algorithm is demonstrated in the numerical example section of this paper.

Effect of Measurement Noise

Noise in the pulse response measurements will result in inaccuracy in the H matrix, and this inaccuracy will lead to error in the predicted outputs at the end of the control task. This section examines the relationship between measurement noise and error in the predicted outputs at the end of the control history.

For a single input system, the predicted outputs at $t_f = n\Delta t$ are given by

$$\hat{y}(n\Delta t) = \hat{H}\boldsymbol{u} = \sum_{i=1}^{n} \hat{\boldsymbol{h}}_i u_i \tag{18}$$

if $\hat{H}$ contains pulse response measurements that are subject to noise. An error vector can be defined as

$$\boldsymbol{e}_i = \hat{\boldsymbol{h}}_i - \boldsymbol{h}_i \tag{19}$$

where $\boldsymbol{h}_i$ is a measurement vector without noise. The expected outer product between two error vectors is

$$E[\boldsymbol{e}_i\boldsymbol{e}_j^T] = K_e\delta_{ij} \tag{20}$$

if the error in measurements is zero-mean and uncorrelated in time, where K_e is the error vector covariance matrix and δ_{ij} is the Kronecker delta. In the limit as Δt approaches zero, the pulse response of the distributed system is proportional to Δt, because the impulse associated with a pulse of unit amplitude and duration Δt is proportional to Δt. If measurement hardware is selected appropriately for the signal levels encountered, the mean square error can be expected to be proportional to the mean square signal in the pulse response measurements. This will make K_e proportional to $(\Delta t)^2$, or $1/n^2$ if $t_f = n\Delta t$ is held constant.

The mean or expected value of $\hat{y}(n\Delta t)$ is given by

$$\mu_y \equiv E[\hat{y}(n\Delta t)] = \sum_{i=1}^{n} E[\hat{\boldsymbol{h}}_i]u_i = H\boldsymbol{u} = y(n\Delta t) \tag{21}$$

if $\boldsymbol{e}_i$ is zero-mean. The covariance matrix for $\hat{y}(n\Delta t)$ is

$$\begin{aligned} K_y &\equiv E[\hat{y}(n\Delta t) - \mu_y)[\hat{y}(n\Delta t) - \mu_y)^T] \\ &= E\left[\sum_{i=1}^{n} \boldsymbol{e}_i u_i \sum_{j=1}^{n} \boldsymbol{e}_j^T u_j\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} u_i u_j K_e \delta_{ij} \le nB^2 K_e \end{aligned} \tag{22}$$

where B is an input bound. Since K_e is proportional to $1/n^2$, K_y is proportional to $1/n$. This result reflects the averaging of error that takes place in the convolution sum in the PRBC method.

For clarity, the process of obtaining the H matrix has been assumed to involve applying only a single pulse in each control input. It should be pointed out that in practice, it will be more advisable to use a string of random pulses in each control input, for example, along with an appropriate identification algorithm, to generate an H matrix that is less sensitive to measurement noise.

If the pulse response measurements are corrupted by noise, a control history $\boldsymbol{u}$ that drives the system to the desired final state will not satisfy the equations $\bar{H}\boldsymbol{u} = \bar{y}_f$, where $\bar{H}$ contains noisy data. For this reason, it may be advisable, for example, to replace the equality constraints in the algorithm of the preceding section with pairs of inequality constraints that require the predicted outputs to be within some tolerance of the desired outputs, where this tolerance would be based on the measurement noise statistics.

Numerical Example

In this section, PRBC is used to solve the minimum time control problem for transverse rest-to-rest translation of a slender beam using bounded force inputs at its two ends. The exact solution of this minimum time control problem is not known. Although PRBC can be used on any type of structure with linear behavior, a uniform beam is used for this example because the infinity of modes are known. This means that modal truncation effects can be virtually eliminated in simulation, and PRBC's effectiveness on a structure having an infinity of modes can be examined.

The motion of the beam is governed by the equation[16]

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + m\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \qquad -L/2 < x < L/2 \tag{23}$$

in which $w(x, t)$ is the transverse displacement, EI is the flexural rigidity, m is the mass per unit length, and the length of the beam is L. For this problem it is convenient to locate the origin at the center of the beam. Shear deformation and rotatory inertia are neglected. There is no moment applied at the ends of the beam, and the beam is controlled by transverse force inputs at its ends, so the boundary conditions are

$$EI\frac{\partial^2 w(x,t)}{\partial x^2}\bigg|_{x=\pm L/2} = 0 \qquad EI\frac{\partial^3 w(x,t)}{\partial x^3}\bigg|_{x=-L/2} = F_1(t)$$
$$EI\frac{\partial^3 w(x,t)}{\partial x^3}\bigg|_{x=L/2} = -F_2(t) \tag{24}$$

The control task consists of translating the beam a distance w_0 as quickly as possible, with no residual vibration at the end of the control task. Hence the initial and final conditions are

$$w(x,0) = \dot{w}(x,0) = \dot{w}(x,t_f) = 0$$
$$w(x,t_f) = w_0 \qquad -L/2 < x < L/2 \tag{25}$$

For this example, it is assumed that there are displacement and velocity sensors at the ends of the beam. Because of the symmetry of the system and the control task, it is sufficient to consider outputs from only one end of the beam. For the same reason, the force inputs are taken to be equal, i.e.,

$$F(t) \equiv F_1(t) = F_2(t) \tag{26}$$

The force inputs are taken to be nonideal, with their dynamic behavior governed by the first-order ordinary differential equation

$$\dot{F}(t) + aF(t) = u(t) \tag{27}$$

in which $u(t)$ is the force command, and $1/a$ is the time constant for the actuator's response. A piecewise constant approximation of the minimum time profile in $u(t)$ is to be obtained using PRBC, with $u(t)$ subject to the bounds $|u(t)| \leq B$.

A modal simulation is used to generate pulse response data for the system, with modal truncation effects reduced to a negligible level by including as many as one hundred symmetric modes in the model. The natural modes of vibration form an orthonormal basis for the response of the system, so the displacement $w(x, t)$ can be represented in terms of the eigenfunctions $\phi_r(x)$ and the modal displacements $\eta_r(t)$ as

$$w(x, t) = \sum_{r=0}^{\infty} \phi_r(x)\eta_r(t) \tag{28}$$

Because of symmetry, the antisymmetric modes need not be included in the analysis. The mass normalized symmetric rigid body and flexible modes are

$$\phi_0(x) = \frac{1}{\sqrt{mL}}$$

$$\phi_r(x) = \sqrt{\frac{2}{mL}} \left[\frac{\cosh(\beta_r L/2)\cos\beta_r x + \cos(\beta_r L/2)\cosh\beta_r x}{[\cosh^2(\beta_r L/2) + \cos^2(\beta_r L/2)]^{1/2}} \right]$$

$$r = 1, 2, \ldots \tag{29}$$

where the β_rs are from those roots of the characteristic equation $\cos\beta_r L \cosh\beta_r L = 1$ that are associated with the symmetric eigenfunctions. The pulse response matrix H is generated by exploiting the orthogonality properties of the modes to obtain modal equations of motion, solving for modal responses to actuator forces resulting from a square pulse in $u(t)$, and adding up the modal contributions to the outputs.

The statement of the minimum time problem is complete when the bound B on the control inputs is specified. For this example, this bound is chosen based on the time $(t_f)_{\text{RIGID}}$ that would be required to carry out the desired translation on a rigid bar having the same mass, with ideal inputs. The relationship between $(t_f)_{\text{RIGID}}$ and the ratio B/w_0 is expressed in the equation

$$\frac{B}{w_0} = \frac{2mL}{(t_f)^2_{\text{RIGID}}} \tag{30}$$

If the bang-bang minimum time solution for the rigid system is applied to the flexible system, the residual energy at the end of the control history will depend upon how $(t_f)_{\text{RIGID}}$ compares to the period of the first flexible mode T. For this example, the ratio B/w_0 is chosen so that $(t_f)_{\text{RIGID}}$ is equal to $T/4$, which indicates that a bang-bang control profile would result in a very large amount of residual vibration. Note that the minimum time for control of a uniform second-order one-dimensional system with unbounded inputs at its two ends is equal to one half of the period of the first flexible mode.[13] The minimum time problem to be solved, then, is for extremely rapid control of the beam.

If the final rigid body displacement is specified in $\bar{y}_f$, it will be obtained exactly, and the accuracy of the final state of the system can be measured in terms of the residual energy at the end of the control task, which is given by the modal sum

$$E_{\text{res}} = \frac{1}{2}\sum_{r=0}^{\infty} [\dot{\eta}_r^2(t_f) + \lambda_r \eta_r^2(t_f)] \tag{31}$$

where $\lambda_r = \omega_r^2$ is an eigenvalue and the square of a natural frequency. In the results presented here, the residual energy is scaled by the maximum energy in the corresponding rigid system in its minimum time control, which occurs at midmaneuver and is given by

$$E_{m-m} = \frac{2mLw_0^2}{(t_f)^2_{\text{RIGID}}} \tag{32}$$

Residual energy is not ordinarily calculated in PRBC because of the absence of a model, but it is calculated for this example and presented as a measure of success in eliminating residual vibration.

For this example, the time constant for the force inputs is set equal to one percent of the first period T. Pulse response data is generated by modal simulation for a time step size Δt equal to one eightieth of $(t_f)_{\text{RIGID}}$, and then H matrices for pulses of width $2\Delta t$, $4\Delta t$, and $8\Delta t$ are obtained as described earlier. The vector $\bar{y}_f$ initially contains only the displacement and velocity in the translational rigid body mode, and then contains the transverse displacement and velocity at one end of the beam at t_f and later time steps. Figure 1 shows how t_f increases as the number of time steps for which outputs are specified increases, and how the ratio of output error norms R decreases. The number of time steps for which outputs are specified is increased until $R \leq 0.002$, at which point the time step size is decreased from $8\Delta t$ to $4\Delta t$. For this example, the time step size is halved whenever R is reduced to 0.002 with the current step size, and the algorithm is terminated when this value is reached with the step size Δt. In all cases, R is obtained using outputs from five time steps spaced $8\Delta t$ apart.

The time required for control t_f increases from its initial value, which is slightly greater than $(t_f)_{\text{RIGID}}$, to a value that is somewhat higher than the final value obtained, and the time step size is $8\Delta t$. Each time the time step size is halved, there is a reduction in t_f due to the improved resolution of the control profile. The final value of t_f is equal to $0.628125T$, which is

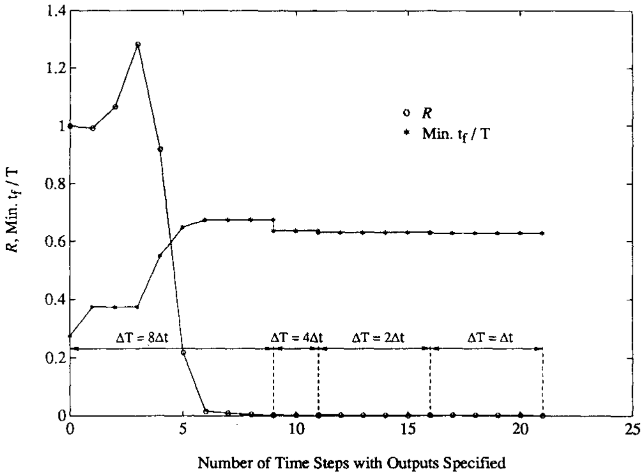

Fig. 1 Variation of output error norm ratio R and final time t_f as the number of time steps with outputs specified increases.

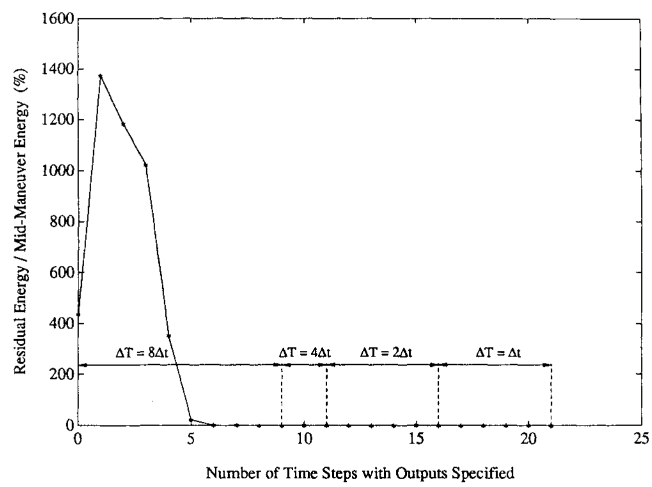

Fig. 2 Variation of residual energy as the number of time steps with outputs specified increases.

about two and one-half times $(t_f)_{\mathrm{RIGID}}$. There are about two hundred time steps in the final control profile, and about two hundred unknowns, when the time step size Δt is used.

Figure 2 shows the residual energy in the system at the end of the control history as a function of the number of time steps for which outputs are specified. The minimum time control for achieving the desired final rigid body mode states results in residual energy more than four times as great as the kinetic energy in the corresponding rigid system at midmaneuver. Initially, as outputs are specified for time steps at the end of the control task, the residual energy increases. This happens because the number of specified outputs is not yet large enough to ensure that the control profile drives the system close to the desired final state. This is evident from Fig. 1, in the values of R and t_f. However, once outputs have been specified for six time steps, the residual energy and R drop close to zero, and t_f levels off at the largest value encountered in the calculations. Once outputs have been specified for eight time steps, the residual energy is driven to less than 0.1% of the midmaneuver energy and stays at approximately that level for the rest of the calculations. This result indicates that the main benefit resulting from continuing the computation with smaller time steps is a reduction in t_f, since the desired final state is already obtained very accurately in this early approximation of the minimum time control profile.

Figure 3 shows the final profiles obtained for the command $u(t)$ and the actuator force $F(t)$. It is obvious that these profiles are not small perturbations of the bang-bang minimum time control for the corresponding rigid system. The

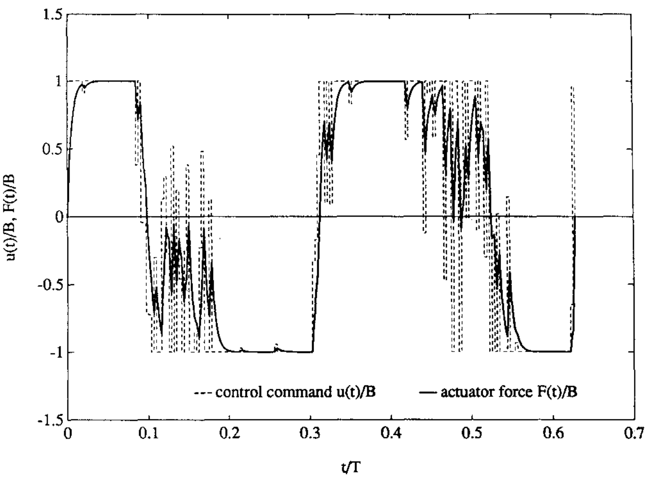

Fig. 3 Minimum time command and actuator force profiles.

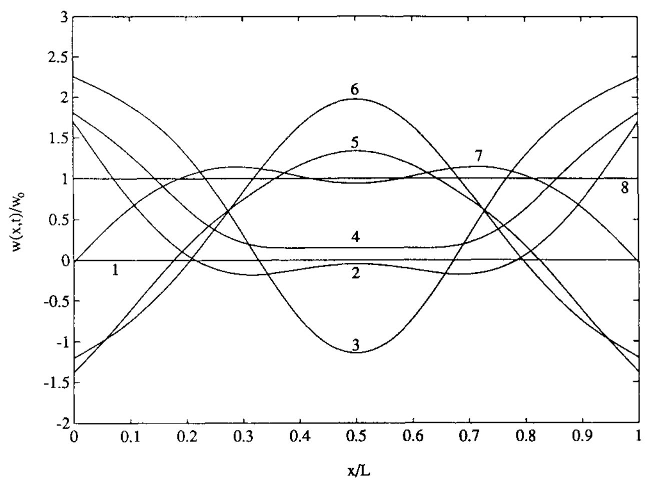

Fig. 4 Beam deflection at several times during control history.

Table 1 CPU time for each time step size

Time step size ΔT	Number of steps in control history	CPU time,[a] s
$8\Delta t$	27	3.15
$4\Delta t$	51	1.84
$2\Delta t$	101	5.05
Δt	201	14.49
Total CPU time		24.53

[a]CPU time for Sun SPARCstation 2.

PRBC approach is not subject to the inherent limitations of a perturbation approach on the rapidity of the control relative to the flexibility of the system. The profiles are not smooth, but the residual energy is negligible. This is consistent with the only known exact solution of a minimum time control problem for a distributed parameter system,[13] and points out an important difference between PRBC and the more conventional modal approach. When the response in a finite number of modes is considered, it is often helpful to smooth the control profile to reduce unwanted excitation of higher modes. However, since there is no modal truncation in the representation of the system dynamics in PRBC, this artificial smoothing is unnecessary. A final observation related to the control profile is that if a finite time constant characterizes the actuator dynamics, little is gained by using a time step size that is much smaller than the time constant.

Figure 4 shows plots of the deflected shape of the beam at intervals of $0.375(t_f)_{\mathrm{RIGID}}$. For this problem, in which the distance translated is small relative to the input force available, for minimum time control the actuators must initially move the ends of the beam well past their final position and excite quite a bit of flexible response. However, this flexible response is eliminated by the time the control task is completed.

Table 1 shows how much computation time was spent approximating the minimum time control profile with each time step size ΔT. The number of unknowns in the control history given in the table is for the final result obtained with each time step size. From the table it is evident that in the computation done with the largest time step size, the algorithm uses only a small fraction of the total computation time to obtain a reasonably accurate approximation of t_f and the minimum time control profiles for the control task, as is evident from Figs. 1 and 2. Once these results have been obtained, the amount of computation that must be done with smaller time steps and more unknowns is quite modest. If the existence of feasible control profiles with final times ranging from $(t_f)_{\mathrm{RIGID}}$ to the minimum t_f for the flexible problem were to be investigated with the time step size Δt, much more computation would be required. The total computation time is less than 25 s on a Sun SPARCstation 2, which is not particularly fast for floating-point computation compared with other workstations in current use.

A minimum norm control profile with the same final time t_f can be obtained if desired. For this problem, solving the associated quadratic programming problem requires about 75 s of additional computation time, and results in a control profile that does not differ substantially from the one shown in Fig. 3.

The quality of the results obtained using the PRBC algorithm is evident from the fact that it is known that no control history with the time step size Δt that satisfies the equality constraints on $\bar{y}_f$ exists with fewer time steps. Also, the time step is small enough that the control profile is divided into more than 200 time steps, and Δt is less than one-third as long as the time constant for the control inputs. Finally, the residual energy in the system is extremely small, indicating that the desired final state is indeed obtained quite accurately.

PRBC has also been applied to the minimum time control problem for a one-dimensional second-order system for which the exact solution is known, and the results are presented in Ref. 17. Since this solution is piecewise constant, it can be obtained exactly using PRBC. Control inputs are located at the two ends of the system, and outputs are taken to be from displacement and velocity sensors at one end of the system, which is symmetric. With the final rigid body mode states specified in $\bar{y}_f$, and with a time step size of one-twentieth of the period of the lowest flexible mode, the number of time steps for which outputs must be specified to obtain the exact minimum time solution varies from zero to eight for different values of the ratio of the distance translated to the bound on control inputs.

The results presented for this example are obtained by PRBC without an explicit model of the system, since the only information that is used in finding the control profile is the pulse response data and the total mass of the system. In spite of this, a very accurate approximate solution of the minimum time problem is obtained, with far less computation than would be required by any other method that takes many modes into consideration. The solution obtained is for extremely rapid control, with a considerable amount of excitation and subsequent elimination of flexible response. Although the PRBC method is intended for systems for which an accurate model is not available, the example presented here demonstrates that it also provides an economical means for studying minimum time control of distributed systems whose properties are known accurately.

Conclusions

In this paper, a powerful new method for solving minimum time control problems for flexible structures is presented. The pulse response based control method does not require an explicit model of the dynamic system to be controlled. Instead, it uses readily obtained measurements of response to pulses in control inputs to obtain minimum time control profiles. Since all modes contribute to the pulse response measurements, there is no modal truncation in the representation of the system dynamics. Actuator dynamics are automatically taken into account as part of the system dynamics in this approach if the test pulses are applied in commands to the control inputs rather than in actual control forces or moments. The precision with which the final state of the system can be specified using pulse response based control is only limited by the observability of the system with the given set of outputs. The effect of measurement noise on the accuracy of the results obtained is shown to decrease as the pulse width decreases, due to the averaging of error that takes place in the convolution sum used in pulse response based control. The control profile obtained using pulse response based control is not bang-bang, which is appropriate since the exact minimum time control profile for the distributed parameter system is not ordinarily bang-bang. A special algorithm is developed for implementing pulse response based control in which pulse response data for several larger pulse widths, or time step sizes, is synthesized from the original pulse response measurement data. Then approximations of the minimum time solution are obtained first with the largest time step size and then with decreasing time step sizes until the pulse width used to obtain the original pulse response data is reached. A numerical example demonstrates the efficiency of the algorithm and the effectiveness of the pulse response based control method. With a modest amount of computation, the algorithm obtains an approximation of the minimum time control profile with a very small time step size and establishes that the number of time steps is minimal. The desired final state of the system is obtained accurately, so an excellent approximation of the minimum time solution is economically obtained.

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